

# On deformation of Poisson manifolds of hydrodynamic type \*

Luca Degiovanni<sup>1</sup>      Franco Magri<sup>2</sup>      Vincenzo Sciacca<sup>3</sup>

1. Dottorato in Matematica, University of Torino  
via C. Alberto 10, 10123 Torino  
*email:* degio@dm.unito.it
2. Department of Mathematics and Applications, University of Milano Bicocca  
via degli Arcimboldi 8, 20126 Milano  
*email:* magri@matapp.unimib.it
3. Dottorato in Matematica, University of Palermo  
via Archirafi 34, 90123 Palermo  
*email:* sciacca@dipmat.math.unipa.it

## Abstract

We study a class of deformations of infinite-dimensional Poisson manifolds of hydrodynamic type which are of interest in the theory of Frobenius manifolds. We prove two results. First, we show that the second cohomology group of these manifolds, in the Poisson-Lichnerowicz cohomology, is “essentially” trivial. Then, we prove a conjecture of B. Dubrovin about the triviality of homogeneous formal deformations of the above manifolds.

## 1 Dubrovin’s conjecture

In this paper we solve a problem proposed by B. Dubrovin in the framework of the theory of Frobenius manifolds [2]. It concerns the deformations of Poisson tensors of hydrodynamic type. The challenge is to show that a large class of these deformations are trivial.

In an epitomized form the problem can be stated as follows. Let  $M$  be a Poisson manifold endowed with a Poisson bivector  $P_0$  fulfilling the Jacobi condition

$$[P_0, P_0] = 0$$

with respect to the Schouten bracket on the algebra of multivector fields on  $M$ . A deformation of  $P_0$  is a formal series

$$P_\epsilon = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \cdots$$

in the space of bivector fields on  $M$  satisfying the Jacobi condition

$$[P_\epsilon, P_\epsilon] = 0 \tag{1}$$

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\*Work sponsored by the Italian Ministry of Research under the project 40%: *Geometry of Integrable Systems*

for any value of the parameter  $\epsilon$ . The deformation is trivial if there exists a formal diffeomorphism  $\phi_\epsilon : M \rightarrow M$ , admitting the Taylor expansion

$$\phi_\epsilon = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \dots,$$

which pulls back  $P_\epsilon$  to  $P_0$ :

$$P_\epsilon = \phi_{\epsilon*}(P_0).$$

Assume that the class of deformations  $P_\epsilon$  and of diffeomorphisms  $\phi_\epsilon$  is restricted by a set of additional conditions to be described below. The demand is to prove that every allowed deformation is trivial, and to provide an explicit procedure to construct the trivializing map  $\phi_\epsilon$  in the class of allowed transformations.

In the concrete form suggested by Dubrovin, the manifold  $M$  is very simple but the class of allowed deformations is rather large. That is the source of difficulty of the problem. Indeed, the manifold  $M$  is the space of  $C^\infty$ -maps  $u^a(x)$  from  $S^1$  into  $\mathbb{R}^n$ , and the bivector  $P_0$  is of hydrodynamic type [1]. By using the so-called “flat coordinates”  $u^a$  in  $\mathbb{R}^n$ , it can be written in the simple form

$$P_0 = g^{ab} \frac{d}{dx}$$

where the coefficients  $g^{ab}$  are the entries of a constant, regular, symmetric  $n \times n$  matrix (not necessarily positive definite). The allowed deformations  $P_\epsilon$  are formal series of matrix-valued differential operators. The coefficient  $P_k$  has degree  $k+1$ , and is written in the form

$$P_k = A_0(u) \frac{d^{k+1}}{dx^{k+1}} + A_1(u; u_x) \frac{d^k}{dx^k} + \dots + A_{k+1}(u; u_x, \dots, u_{k+1}).$$

The entries of the matrix coefficient  $A_l$  are assumed to be homogeneous polynomials of degree  $l$  in the derivatives of the field functions  $u^a(x)$ . The degree of a polynomial is computed by attributing degree zero to the field functions, degree one to their first derivatives, degree two to the second derivatives, and so on. By this requirement the form of the operator  $P_k$  is fixed up to the choice of a increasing number of arbitrary functions of the coordinates  $u^a$ . These functions, finally, must be chosen so to guarantee that the operator  $P_k$  is skewsymmetric

$$P_k^* = -P_k \tag{2}$$

and that the Jacobi condition (1) is satisfied at the order  $k$ . This means that the first  $k$  operators  $P_l$  must be chosen so to verify the  $k$  conditions

$$\sum_{i+j=l} [P_i, P_j] = 0 \quad l = 1, \dots, k$$

or, explicitly,

$$\begin{aligned} 2[P_0, P_1] &= 0 \\ 2[P_0, P_2] + [P_1, P_1] &= 0 \\ 2[P_0, P_3] + 2[P_1, P_2] &= 0 \\ &\dots \end{aligned} \tag{3}$$

and so on. The conjecture of Dubrovin is that all these homogeneous deformations are trivial, and that the trivializing map is homogeneous as well.

To better understand the problem, let us consider the scalar-valued case. According to the rules of the game the first three coefficients of the deformations  $P_\epsilon$  have the form

$$\begin{aligned} P_0 &= \frac{d}{dx} \\ P_1 &= A(u) \frac{d^2}{dx^2} + B(u) u_x \frac{d}{dx} \\ &\quad + (C(u) u_{xx} + D(u) u_x^2) \\ P_2 &= E(u) \frac{d^3}{dx^3} + F(u) u_x \frac{d^2}{dx^2} + (G(u) u_{xx} + H(u) u_x^2) \frac{d}{dx} \\ &\quad + (L(u) u_{xxx} + M(u) u_{xx} u_x + N(u) u_x^3). \end{aligned}$$

They depend on eleven arbitrary functions of the coordinate  $u$ . By imposing the skewsymmetry condition (2) this number falls to four. Indeed we get the seven differential constraints

$$\begin{aligned} A &= 0, \quad 2C = B, \quad 2D = B' \\ 2F &= 3E', \quad 4L = 2G - E', \quad 4N = 2H' - E''' \\ 4M &= 2G' + 4H - 3E''. \end{aligned}$$

The remaining four functions are constrained by the Jacobi condition. To work out this condition we use the operator form of the Schouten bracket of two skewsymmetric operators  $P$  and  $Q$ . We need the following notations. We denote by  $Q_u \alpha$  the value of the differential operator  $Q_u$  on the argument  $\alpha$ , and by

$$Q'_u(\alpha; \dot{u}) = \frac{d}{ds} Q_{u+s\dot{u}} \alpha|_{s=0}$$

its derivative along the vector field  $\dot{u}$ . The adjoint of this derivative with respect to  $\dot{u}$  is denoted by  $Q'^*_u(\alpha; \beta)$ . It is defined by

$$\langle Q'_u(\alpha; \dot{u}), \beta \rangle = \langle \dot{u}, Q'^*_u(\alpha; \beta) \rangle,$$

where the pairing between vector fields and 1-forms is defined, as usual, by

$$\langle \dot{u}, \beta \rangle = \int_{S^1} \dot{u}(x) \beta(x) dx.$$

(Of course, in the vector-valued case we have to sum over the different components). Then the Schouten bracket is given by [4]:

$$\begin{aligned} 2[P, Q](\alpha, \beta) &= P'_u(\alpha; Q_u \beta) - P'_u(\beta; Q_u \alpha) - Q_u \cdot P'^*_u(\alpha; \beta) \\ &\quad + Q'_u(\alpha; P_u \beta) - Q'_u(\beta; P_u \alpha) - P_u \cdot Q'^*_u(\alpha; \beta). \end{aligned}$$

In our example, the bivector  $P_0$  is constant. Therefore, the first two Jacobi conditions (3) take the simple form:

$$P'_1(\alpha; P_0 \beta) - P'_1(\beta; P_0 \alpha) - P_0 \cdot P'^*_1(\alpha; \beta) = 0$$

and

$$\begin{aligned} &P'_2(\alpha; P_0 \beta) - P'_2(\beta; P_0 \alpha) - P_0 \cdot P'^*_2(\alpha; \beta) \\ &+ P'_1(\alpha; P_1 \beta) - P'_1(\beta; P_1 \alpha) - P_1 \cdot P'^*_1(\alpha; \beta) = 0 \end{aligned}$$

respectively. By expanding these operator conditions, we obtain two further relations

$$\begin{aligned} B &= 0 \\ 4H &= 2G' + E'' \end{aligned}$$

among the eleven functions  $(A(u), \dots, N(u))$ . Solving them and the previous ones we obtain that the first coefficients of  $P_\epsilon$  are:

$$P_1 = 0 \quad (4)$$

$$\begin{aligned} P_2 &= E(u) \frac{d^3}{dx^3} + \frac{3}{2} E'(u) u_x \frac{d^2}{dx^2} + G(u) u_{xx} \frac{d}{dx} \\ &\quad + \frac{1}{2} (G'(u) + \frac{1}{2} E''(u) u_x^2) \frac{d}{dx} + \frac{1}{2} (G(u) - \frac{1}{2} E'(u)) u_{xxx} \\ &\quad + (G'(u) - \frac{1}{2} E''(u)) u_{xx} u_x + \frac{1}{4} (G''(u) - \frac{1}{2} E'''(u)) u_x^3. \end{aligned} \quad (5)$$

Up to the second order in  $\epsilon$ , this is the most general homogeneous deformation in the scalar case.

To check Dubrovin's conjecture to the second-order in  $\epsilon$ , it is enough to consider the homogeneous map

$$\phi_\epsilon(u) = u + \epsilon R(u) u_x + \epsilon^2 (S(u) u_{xx} + T(u) u_x^2) + \dots$$

and to use the operator form [4]

$$P_\epsilon = \phi'_{\epsilon u} \cdot P_0 \cdot \phi'^*_{\epsilon u} \quad (6)$$

of the transformation law for bivectors. As before,  $\phi'^*_{\epsilon u}$  denotes the adjoint operator of the Fréchet derivative of  $\phi_\epsilon(u)$ . By expanding Eq. (6), we find:

$$\begin{aligned} P_1 &= 0 \\ P_2 &= (2S(u) - R^2(u)) \frac{d^3}{dx^3} + 3(S'(u) - R(u)R'(u)) u_x \frac{d^2}{dx^2} \\ &\quad + (5S'(u) - 4T(u) - R(u)R'(u)) u_{xx} \frac{d}{dx} \\ &\quad + (3S''(u) - 2T'(u) - R(u)R''(u) - R'(u)^2) u_x^2 \frac{d}{dx} \\ &\quad + 2(S'(u) - T(u)) u_{xxx} + 4(S''(u) - T'(u)) u_{xx} u_x \\ &\quad + (S'''(u) - T''(u)) u_x^3. \end{aligned}$$

By comparison with Eq. (4) and Eq. (5), we realize that Dubrovin's conjecture is true in the scalar case, up to the second order in  $\epsilon$ . In fact the choices

$$\begin{aligned} R(u) &= 0 \\ S(u) &= \frac{1}{2} E(u) \\ T(u) &= \frac{5}{8} E'(u) - \frac{1}{4} G(u) \end{aligned} \quad (7)$$

allow to reconstruct the diffeomorphism  $\phi_\epsilon$  from the deformation  $P_\epsilon$ .

The questions now are: what happens at higher order in  $\epsilon$ , or in the matrix case? What is the meaning of the relations (7) connecting  $P_\epsilon$  to  $\phi_\epsilon$ ? Due to the great complexity of the computations, it is clear that any direct attack is beyond our reach. We have to devise an alternative approach. Our strategy is to convert the given problem into a problem in Poisson-Lichnerowicz cohomology. It is based on two remarks:

1. Poisson manifold of hydrodynamic type are transversally constant.
2. The second cohomology group in the Poisson-Lichnerowicz cohomology of these manifolds is “essentially” trivial.

The first remark concerns the symplectic foliation associated with the Poisson bivector  $P_0$ . In our example, this foliation is rather regular. All the leaves are affine hyperplane of codimension  $n$ . They are the level sets of  $n$  globally defined *Casimir functions*  $C^a$ ,  $a = 1, 2, \dots, n$ . Furthermore there exists an abelian group of symplectic diffeomorphisms which transform the symplectic leaves among themselves.

The second remark concerns the bivectors  $Q$  fulfilling the condition

$$[P_0, Q] = 0.$$

They must be compared with the bivectors

$$Q = L_X(P_0)$$

which are Lie derivatives of  $P_0$  along any vector field  $X$  on  $M$ . The former are called 2-cocycles in the Poisson-Lichnerowicz cohomology defined by  $P_0$  on  $M$  [3]. The latter are called 2-coboundaries. Not all cocycles are coboundaries. A first simple obstruction is the vanishing of the Poisson bracket of the Casimir functions  $C^a$  with respect to  $Q$ :

$$\{C^a, C^b\}_Q = 0. \quad (8)$$

Further obstructions depend on the topology of the manifold. The main result of the paper is the proof, in § 3, that these further obstructions are absent on a Poisson manifold of hydrodynamic type. By a combined use of ideas of the theory of transversally constant Poisson manifolds (suitably extended to infinite-dimensional manifolds) and of the operator approach to the inverse problem of the Calculus of Variations in the style of Volterra [9] [8], we show that every 2-cocycle verifying Eq. (8) is a 2-coboundary, and we give an explicit formula for the vector field  $X$  (called the potential of  $Q$ ). Several examples of this result are shown in § 4, where possible applications to the classifications of bihamiltonian manifolds are also briefly discussed.

Once equipped with this result, the conjecture of Dubrovin can be proved in a direct and simple way. First we notice that the homogeneous deformations pass the obstructions (8). Then we notice that the Jacobi condition  $[P_\epsilon, P_\epsilon] = 0$  may be replaced by a recursive system of cohomological equations. This leads to a simple general representation of the deformation  $P_\epsilon$ . The argument goes as follows. Consider the first Jacobi condition

$$[P_0, P_1] = 0.$$

It is already in a cohomological form. By the main result of § 3, it follows that there exists a vector field  $X_1$ , such that

$$P_1 = L_{X_1}(P_0).$$

By inserting this information into the second Jacobi equation

$$2[P_0, P_2] + [P_1, P_1] = 0$$

we get a new cohomological equation

$$\left[ P_0, P_2 - \frac{1}{2} L_{X_1}^2(P_0) \right] = 0.$$

hence there exists a second vector field  $X_2$  such that

$$P_2 = L_{X_2}(P_0) + \frac{1}{2} L_{X_1}^2(P_0).$$

By induction, one proves the existence of a sequence of vector fields  $\{X_k\}_{k \in \mathbb{N}}$  such that all the coefficients  $P_k$  of the deformation  $P_\epsilon$  admits the representation

$$P_k = \sum_{j_1 + 2j_2 + \dots + kj_k = k} \left( \frac{L_{X_k}^{j_k}}{j_k!} \frac{L_{X_{k-1}}^{j_{k-1}}}{j_{k-1}!} \dots \frac{L_{X_1}^{j_1}}{j_1!} \right) (P_0). \quad (9)$$

This result gives a complete control of the deformations of Poisson brackets of hydrodynamic type. In particular it allows to give the following simple proof of the Dubrovin's conjecture. Consider separately the different flows  $\phi_{t_k}^{(k)}$  associated with the vector fields  $X_k$ . Give them a different weight by setting

$$t_k = \epsilon^k,$$

and make the ordered product of these flows by multiplying each flow by the subsequent one on the left. The result is the one-parameter family of diffeomorphism

$$\phi_\epsilon = \prod_{k \geq 1} \phi_{\epsilon^k}^{(k)}.$$

It provides the solution we were looking for. Indeed, according to the theory of "Lie transform", Eq. (9) are equivalent to the transformation law

$$P_\epsilon = \phi_{\epsilon*}(P_0)$$

as required. We believe that the strategy sketched above is of interest in itself, and that it can be profitably used in more general context. In our opinion it can provide, for instance, new insights on the problem of classification of bihamiltonian manifolds associated with soliton equations.

## 2 Transversally constant Poisson manifolds

In this section we collect the few ideas of the theory of Poisson manifolds which are used later on to solve Dubrovin's conjecture. Our interest is mainly centered around the difference between 2-cocycles and 2-coboundaries on a regular transversally constant Poisson manifold.

We recall that a finite-dimensional Poisson manifold  $(M, P)$  is *regular* if the symplectic foliation defined by the Poisson bivector  $P$  has constant rank. Let  $k$  denote the corank of the foliation. It follows that, around any point of the manifold,

there exist  $k$  functions  $C^a$ ,  $a = 1, 2, \dots, k$ , which are independent and constant on each symplectic leaf. They are called Casimir functions. Their differentials  $dC^a$  span the kernel of the bivector  $P$ . We also recall that the Poisson manifold is called *transversally constant* [7] if there exist  $k$  vector fields  $Z_a$  which are transversal to the symplectic leaves and are symmetries of  $P$ :

$$L_{Z^a}(P) = 0.$$

Without loss of generality, one can assume that these vector fields satisfy the normalization conditions

$$Z_a(C^b) = \delta_a^b$$

with respect to the chosen family of Casimir functions.

The local structure of a transversally constant Poisson manifold is quite simple: essentially is the product of a symplectic leaf and of the abelian group generated by the vector fields  $Z_a$ . In particular, the tangent space at any point  $m$  can be split into the direct sum

$$T_m M = H_m \oplus V_m$$

of an “horizontal space”  $H_m$  (the tangent space of the symplectic leaf) and of a “vertical space”  $V_m$ , spanned by the vector fields  $Z_a$ . This splitting induces a corresponding decomposition of the dual space and, hence, of any tensor field on  $M$ . For a bivector  $Q$  the basic elements are the vector fields

$$X^a = QdC^a$$

and the horizontal bivector

$$Q_H = \pi_H \circ Q \circ \pi_H^*$$

where  $\pi_H$  denotes, as usual, the canonical projection on  $H$  along  $V$ . A simple computation gives

$$Q_H = Q + X^a \wedge Z_a + \frac{1}{2} X^a(C^b) Z_a \wedge Z_b. \quad (10)$$

They already contain the clue of the distinction between 2-cocycle and 2-coboundaries.

**Lemma 1** *If  $Q$  is a cocycle the vector fields  $X^a$  are symmetries of  $P$  and  $Q_H$  is a cocycle. If  $Q$  is a coboundary the vector fields  $X^a$  are Hamiltonian and  $Q_H$  is a coboundary.*

*Proof.* If  $Q$  is a cocycle we have

$$L_{QdF}(P) + L_{PdF}(Q) = 0$$

for any function  $F$ . For  $F = C^a$ , this equation shows that  $X^a$  is a symmetry of  $P$ . Hence

$$\left[ P, X^a \wedge Z_a + \frac{1}{2} X^a(C^b) Z_a \wedge Z_b \right] = 0$$

since both  $X^a$  and  $Z_a$  are symmetries of  $P$  and  $X^a(C^b)$  is a Casimir function. This show that

$$[P, Q_H] = 0$$

as claimed.

If  $Q = L_X(P)$  is a coboundary, we find

$$QdC^a = L_X(P)dC^a = L_X(PdC^a) - PdX(C^a) = -PdX(C^a)$$

showing that

$$QdC^a = PdH^a$$

with

$$H^a = -X(C^a).$$

Therefore we find

$$X^a \wedge Z_a + \frac{1}{2}X^a(C^b)Z_a \wedge Z_b = L_Z(P)$$

with

$$Z = -H^a Z_a.$$

This proves the second part of the Lemma.  $\square$

The previous remark alone is sufficient for the later applications. However, in view of adapting the result to the case of infinite-dimensional Poisson manifolds of hydrodynamic type, it is better to restate it in a different form. The idea is to trade multivectors for forms. To this end, we first split the vector fields  $X^a$  into horizontal and vertical parts. Then, the components of the vertical parts are used to define the matrix

$$\{C^a, C^b\}_Q := X^a(C^b).$$

The horizontal parts  $X_H^a$  are, instead, used to define  $k$  1-forms  $\theta^a$  living on the symplectic leaves. They are given by

$$\theta^a(X_F) = X_H^a(F) \tag{11}$$

where  $X_F = PdF$  is the Hamiltonian vector field associated with the function  $F$ . Similarly, the horizontal bivector  $Q_H$  is traded for a 2-form  $\omega$ , living on the symplectic leaves, according to

$$\omega(X_F, X_G) = Q_H(dF, dG).$$

The outcome is that any bivector  $Q$  on a regular transversally constant Poisson manifold  $M$  is characterized by three elements:

1. the functions  $\{C^a, C^b\}_Q$
2. the 1-forms  $\theta^a$
3. the 2-form  $\omega$ .

As a simple restatement of the previous Lemma, we obtain the following result.

**Lemma 2** *If  $Q$  is a cocycle  $\{C^a, C^b\}_Q$  is a Casimir function, and the forms  $\theta^a$  and  $\omega$  are closed. If  $Q$  is a coboundary the functions  $\{C^a, C^b\}_Q$  vanish, and the forms  $\theta^a$  and  $\omega$  are exact.*

We do not give the proof of this result, that can be found in [7]. Instead, for further convenience, we show its converse in the following form.



**Lemma 3** *If the functions  $\{C^a, C^b\}_Q$  vanish,*

$$\{C^a, C^b\}_Q = 0, \quad (12)$$

*and the forms  $\theta^a$  and  $\omega$  are exact,*

$$\theta^a = dH^a \quad (13)$$

$$\omega = d\theta, \quad (14)$$

*the bivector  $Q$  is a coboundary. Its potential  $X$  is given by*

$$X = -H^a Z_a + P\theta. \quad (15)$$

*Proof.* The first assumption (12) entails that the vector fields  $X^a$  are tangent to the symplectic leaves. Hence  $X^a = X_H^a$ . Thus the definition (11) and the second assumption (13) leads to

$$X^a = PdH^a.$$

Set  $Z = -H^a Z_a$ . As in the proof of Lemma 1 we get

$$Q = Q_H + L_Z(P). \quad (16)$$

Finally, we notice that the third assumption (14) entails

$$Q_H(dF, dG) = \omega(X_F, X_G) = d\theta(X_F, X_G) = L_{P\theta}(P)(dF, dG).$$

Hence the previous equation becomes

$$Q = L_{P\theta}(P) + L_Z(P) = L_X(P)$$

as claimed.  $\square$

A difficulty is readily met in trying to extend the previous result to infinite-dimensional manifolds. It is connected to the definition (10) of the bivector  $Q_H$  where the operation of exterior product is used. We have found difficult to extend this formula in the infinite-dimensional setting where vector fields and bivectors are represented by differential operators. To circumvent this difficulty, we can follow a two-steps procedure, where the vector fields  $X^a$  come first, and only later the bivector  $Q_H$  is introduced as the complementary part of  $L_Z(P)$  in the splitting (16) of  $Q$ . This detour leads to an “eight steps algorithm” to check if a given bivector  $Q$  on a transversally constant Poisson manifold is a coboundary. They are:

1. Check that the functions  $\{C^a, C^b\}_Q$  vanish.
2. Check that the vector fields  $QdC^a$  are Hamiltonian:  $QdC^a = PdH^a$ .
3. Introduce the transversal vector field  $Z = -H^a Z_a$ .
4. Compute the Lie derivatives of  $P$  along  $Z$ .
5. Define the horizontal bivector  $Q_H$  according to:  $Q_H = Q - L_Z(P)$ .
6. Introduce the 2-form  $\omega$  by factorizing  $Q_H$  according to:  $Q_H = P \circ \omega \circ P$ .
7. Check that this form is exact on the symplectic leaves.
8. Compute its potential  $\theta$ .

At the end of this long chain of tests, one can affirm that  $Q$  is a coboundary and construct its potential  $X$  according to Eq. (15). In the next section we shall display this procedure for manifolds of hydrodynamic type.

### 3 Poisson manifolds of hydrodynamic type

Let now

$$P = P_0 = g^{ab} \frac{d}{dx}.$$

We notice that this bivector admits  $k$  globally defined Casimir functions

$$C^a(u) = \int_0^1 u^a(x) dx.$$

Therefore its symplectic leaves are affine hyperplanes and the manifold is regular. We also notice that the vector fields

$$Z_a : \quad \dot{u}^b = \delta_a^b$$

are globally defined transversal symmetries. Hence, the manifold is transversally constant as well.

On this manifold we consider the class of bivectors  $Q$  which are represented by matrix-valued differential operators

$$Q = \sum_{k \geq 0} A_k(u, u_x, \dots) \frac{d^k}{dx^k} \quad (17)$$

and which satisfy the simple condition

$$\{C^a, C^b\}_Q = 0. \quad (18)$$

We stress that no homogeneity conditions are imposed on  $Q$ . So the present class of bivectors is bigger than that considered in Dubrovin's conjecture. We shall prove

**Proposition 1** *Each cocycle  $Q$  in this class is a coboundary.*

To appreciate the strength of this result, let us consider the case of a single loop function  $u(x)$ . Condition (18) is automatically verified in this case, since there is only one Casimir function, and therefore we can conclude that *every* scalar-valued cocycle is a coboundary. This result is far from being trivial. Let us check this claim for the simple cocycle  $P_2$  considered in § 1. We have to exhibit a vector field

$$\dot{u} = X(u, u_x, u_{xx})$$

such that

$$-P_2 = -L_X(P_0) = X'_u \cdot P_0 + P_0 \cdot X'^{*}_u$$

where  $X'^{*}_u$  is the formal adjoint of the Fréchet derivative  $X'_u$  of the operator defining the vector field  $X$ . A reasonable guess is to look for a homogeneous vector field

$$X = a(u)u_{xx} + b(u)u_x^2.$$

Since

$$X'_u = a(u) \frac{d^2}{dx^2} + 2b(u)u_x \frac{d}{dx} + a'(u)u_{xx} + b'(u)u_x^2,$$

a simple computation leads to

$$\begin{aligned}
-L_X(P_0) = & 2a(u)\frac{d^3}{dx^3} + 3a'(u)u_x\frac{d^2}{dx^2} + (5a'(u) - 4b(u))u_{xx}\frac{d}{dx} \\
& + (3a''(u) - 2b'(u))u_x^2\frac{d}{dx} + 2(a'(u) - b(u))u_xxx \\
& + 4(a''(u) - b'(u))u_xu_{xx} + (a'''(u) - b''(u))u_x^3.
\end{aligned}$$

The problem is solved by noticing that the relations

$$a(u) = -\frac{1}{2}E(u)$$

$$b(u) = \frac{1}{4}G(u) - \frac{5}{8}E'(u)$$

allow to identify the operator  $L_X(P)$  with  $P_2$ , for *any* choice of the function  $E(u)$  and  $G(u)$  (see Eq. (5)), as claimed in Proposition 1.

In this section we shall show that the above relations are simply an instance of the general formula (15), defining the potential  $X$  of any coboundary of a transversally constant Poisson manifold. The main difficulty is to identify the geometrical objects (the vector fields  $QdC^a$ , the 1-forms  $\theta^a$ , and the 2-form  $\omega$ ) to be associated with each bivector of the form (17). To this end, it is useful to split the operator  $Q$  into the sum of three operators. The first operator has degree zero. Therefore it is simply a skewsymmetric matrix  $E$ , whose entries are functions of the loops  $u^a(x)$  and of their derivatives. The second operator has order one. It is written as the anticommutator  $S \cdot \frac{d}{dx} + \frac{d}{dx} \cdot S$  of a symmetric matrix  $S$  with  $\frac{d}{dx}$ . The third operator, finally, collect all the higher order terms.

**Lemma 4** *Any bivector  $Q$  can be uniquely written in the form:*

$$Q = E + S \cdot \frac{d}{dx} + \frac{d}{dx} \cdot S + \frac{d}{dx} \cdot \Lambda \cdot \frac{d}{dx} \quad (19)$$

where  $\Lambda$  is the skewsymmetric operator

$$\Lambda = \sum_{k \geq 0} \left( \Lambda_k \cdot \frac{d^k}{dx^k} + \frac{d^k}{dx^k} \cdot \Lambda_k \right).$$

The coefficients  $\Lambda_k$  of this operator are alternatively symmetric and skewsymmetric matrices, according to the order of the derivatives.

This lemma is very simple to prove, but it is interesting because each term in the splitting (19) has a geometrical meaning. Roughly speaking, the first term  $E$  controls the brackets  $\{C^a, C^b\}_Q$ , the second term controls the 1-forms  $\theta^a$ , and the third term controls the 2-form  $\omega$ . By using this representation formula we can now work out the “eight steps algorithm” stated at the end of the previous section.

**Step 1:** *the vanishing of the functions  $\{C^a, C^b\}_Q$ .*

Since the differentials of the Casimir functions are the constant matrices

$$\frac{\delta C^a}{\delta u^b} = \delta_b^a$$

we easily find

$$\{C^a, C^b\}_Q = \int_0^1 E^{ab} dx$$

where  $E^{ab}$  is the entry of place  $(a, b)$  in the matrix  $E$ . Therefore, condition (18) holds iff there exists a second skewsymmetric matrix  $\mathcal{E}$  such that

$$E = \frac{d}{dx}(\mathcal{E}).$$

Writing this condition in the commutator form

$$E = \frac{d}{dx} \cdot \mathcal{E} - \mathcal{E} \cdot \frac{d}{dx}$$

we can easily eliminate  $E$  from the representation (19) of  $Q$ . Setting  $B = \mathcal{E} + S$  we get

$$Q = B^t \cdot \frac{d}{dx} + \frac{d}{dx} \cdot B + \frac{d}{dx} \cdot \Lambda \cdot \frac{d}{dx}.$$

Finally we replace the differential operator  $\frac{d}{dx}$  by the Poisson bivector

$$P = G \cdot \frac{d}{dx}.$$

We then arrive to the following useful second representation theorem.

**Lemma 5** *Each bivector  $Q$  for which  $\{C^a, C^b\}_Q = 0$  can be uniquely represented in the form*

$$Q = A^t \cdot P + P \cdot A + P \cdot \Gamma \cdot P \quad (20)$$

where  $A$  is the  $n \times n$  matrix and  $\Gamma$  is the skewsymmetric differential operator given by:

$$\begin{aligned} B &= G \cdot A \\ \Lambda &= G \cdot \Gamma \cdot G. \end{aligned}$$

**Step 2:** *the vector fields  $QdC^a$  are Hamiltonian.*

Since  $\{C^a, C^b\}_Q = 0$  we know that the vector fields  $QdC^a$  are tangent to the symplectic leaves of  $P$ . Therefore there exist 1-forms  $\theta^a$  such that

$$QdC^a = P\theta^a.$$

From the representation theorem we easily recognize that the 1-form  $\theta^a$  is given by the  $a$ -th column of the matrix  $A$ . So the component  $b$  of the 1-form  $\theta^a$  is the entry  $A_b^a$  of place  $(a, b)$  of the matrix  $A$ :

$$\theta_b^a = A_b^a.$$

We further know that the vector fields  $QdC^a$  are symmetries of  $P$  by the cocycle condition. If we work out explicitly the condition

$$L_{QdC^a}(P) = 0 \quad (21)$$

in the operator formalism we have

$$\begin{aligned} -L_{P\theta^a}(P) &= P \cdot \theta^{a'} \cdot P - P \cdot \theta^{a'*} \cdot P \\ &= P \cdot (\theta^{a'} - \theta^{a'*}) \cdot P = 0. \end{aligned}$$

This is the same as writing

$$\frac{d}{dx} \cdot (\theta^{a'} - \theta^{a'*}) \cdot \frac{d}{dx} = 0. \quad (22)$$

Let us expand the differential operator  $\theta^{a'} - \theta^{a'*}$  in power of  $\frac{d}{dx}$ :

$$\theta^{a'} - \theta^{a'*} = A_0 + A_1 \cdot \frac{d}{dx} + \cdots + A_n \cdot \frac{d}{dx^n}.$$

Substituting into the previous equation we obtain

$$\begin{aligned} \frac{d}{dx} \cdot (\theta^{a'} - \theta^{a'*}) \cdot \frac{d}{dx} &= A_{0x} \cdot \frac{d}{dx} + (A_0 + A_{1x}) \cdot \frac{d^2}{dx^2} + \cdots \\ &\quad + (A_{n-1} + A_{nx}) \cdot \frac{d^{n+1}}{dx^{n+1}} + A_n \cdot \frac{d^{n+2}}{dx^{n+2}} \end{aligned}$$

showing that the condition (22) can be verified iff  $\theta^{a'} - \theta^{a'*} = 0$ . This means that the Fréchet derivative of the operator  $\theta^a$  is symmetric and, therefore, that this operator is potential [6]. In geometric language this means that the 1-form  $\theta^a$  is closed and therefore exact, since the topology of the manifold  $M$  is simple. The potential is the functional

$$H^a = \int_0^1 h^a(u, u_x, \dots) dx$$

where, according to [5],

$$h^a = \int_0^1 A_b^a(\lambda u, \lambda u_x, \dots) u^b d\lambda. \quad (23)$$

We have thus proved that the vector fields  $QdC^a$  are Hamiltonian.

**Step 3:** *the transversal vector field  $Z$ .*

We choose the transversal vector field

$$Z = -h^a(u, u_x, \dots) Z_a \quad (24)$$

(sum over repeated index  $a$ ). We notice that by this choice we depart slightly from the geometrical scheme. According to the third step of § 2 we should have introduced at this point the vector field

$$\hat{Z} = - \left( \int_0^1 h^a(u, u_x, \dots) dx \right) Z_a$$

whose components are the functionals  $H^a$ , instead of the associated densities  $h^a$ . The change is permitted since  $Z(C^a) = \hat{Z}(C^a)$  and so the functions  $C^a$  are Casimir

functions also for  $Q - L_Z P$ . This fact allows to still define a 2-form  $\omega$  but in general this is different from the one associated with the previous “horizontal” part of the bivector  $Q$ . Our choice has the advantage that the vector field  $Z$  is local.

**Step 4:** *the Lie derivative  $L_Z(P)$ .*

The next step is to compute the Lie derivative of  $P$  along  $Z$ . In the operator formalism this is easily accomplished if we know the Fréchet derivative  $Z'_u$  of the vector field  $Z$ . This derivative is a matrix differential operator. A key property is that the zero-order term of this operator is the transpose of the matrix  $A$  defining the 1-form  $\theta^a$ .

**Lemma 6** *The Fréchet derivative  $Z'$  of the vector field  $Z$  may be uniquely represented as the difference*

$$Z' = -A^\dagger + P \cdot R \quad (25)$$

*of the transpose of the matrix  $A$  and of a factorized differential operator  $P \cdot R$ , taking into account all the higher-order terms appearing in  $Z'$ .*

*Proof.* The identity (25) is nothing else than a disguised form of the Lagrange identity

$$(\alpha, Z'_u \phi) - (\phi, Z'^*_u \alpha) = \frac{d}{dx} B(\alpha, \phi) \quad (26)$$

used to define the formal adjoint of the operator  $Z'$ . In this identity  $\alpha$  and  $\phi$  are arbitrary, and the bracket denotes the usual scalar product in  $\mathbb{R}^n$ . We notice that, by Eq. (24) the vector  $Z'^*_u(e_l)$  is the opposite of the Euler operator associated with the lagrangian density  $h^l$ ,

$$Z'^*_u(e_l) = -\frac{\delta h^l}{\delta u}$$

and we write the identity (26), for  $\alpha = e_l$ , in the operator form

$$-h^{l'}(\phi) + \sum_{b=1}^n \phi^b \frac{\delta h^l}{\delta u^b} = \frac{d}{dx} B(e_l, \phi)$$

where  $h^{l'}$  is the Fréchet derivative of the scalar differential operator  $h^l$ . One easily recognizes in this equation the identity (25) by recalling that

$$A_b^l = \frac{\delta h^l}{\delta u^b}.$$

□

The identity (25) allows to perform the fourth step in our program rather easily. By using once the operator form of the Lie derivative of  $P$  we obtain

$$\begin{aligned} L_Z(P) &= -Z' \cdot P - P \cdot Z'^* \\ &= A^\dagger \cdot P + P \cdot A + P \cdot (R^* - R) \cdot P. \end{aligned}$$

**Steps 5 and 6:** *the horizontal bivector  $Q_H$  and the 2-form  $\omega$ .*

By subtracting this identity from the basic representation formula (20), we obtain

$$\begin{aligned} Q &= L_Z(P) + P \cdot (\Gamma + R - R^*) \cdot P \\ &= L_Z(P) + P \cdot \Omega \cdot P. \end{aligned} \quad (27)$$

It allows to identify the 2-form  $\omega$  with the restriction to the symplectic leaves of  $P$  of the differential operator  $-\Omega$ , where

$$\Omega = \Gamma + R - R^*$$

defined on  $M$ . The explicit computation of this form is algorithmic, as shown by the examples given in the next section. We can thus conclude that we have a systematic procedure to identify the 2-form  $\omega$ .

**Step 7:** *the 2-form  $\omega$  is exact.*

The last steps are now performed along a well-established path. The closure of the 2-form  $\Omega$  follows from the cocycle condition  $[P, Q] = 0$ . By using the operators form of this condition we obtain:

$$\begin{aligned} [P, Q](\alpha, \beta, \gamma) &= [P, P \cdot \Omega \cdot P](\alpha, \beta, \gamma) \\ &= \langle \alpha, P \cdot \Omega'_u(P\beta; P\gamma) \rangle + \langle \beta, P \cdot \Omega'_u(P\gamma; P\alpha) \rangle + \langle \gamma, P \cdot \Omega'_u(P\alpha; P\beta) \rangle \\ &= \langle \alpha, P \cdot [\Omega'_u(P\beta; P\gamma) - \Omega'_u(P\gamma; P\beta) + \Omega'_u{}^*(P\beta; P\gamma)] \rangle \\ &= 0. \end{aligned}$$

Therefore  $Q$  is a cocycle iff  $\Omega$  verifies the equation

$$P \cdot [\Omega'_u(P\beta; P\gamma) - \Omega'_u(P\gamma; P\beta) + \Omega'_u{}^*(P\beta; P\gamma)] = 0 \quad (28)$$

for any choice of the arguments  $\beta$  and  $\gamma$ . Let us fix  $\beta$ . We can regard the previous equation as a differential equation on  $\gamma$  of the form

$$\left(\frac{d}{dx} \cdot T \cdot \frac{d}{dx}\right) \gamma = 0$$

where  $T$  is a suitable differential operator depending on  $\beta$ . By the argument already used in discussing equation (22) we see that this equation can be verified by any  $\gamma$  only if  $T = 0$ . This give rise to a new differential system on  $\beta$ . Once again it can be satisfied by any  $\beta$  only if the equations are identically vanishing. Thus we conclude that the operator Eq. (28) holds iff

$$\Omega'_u(\phi; \psi) - \Omega'_u(\psi; \phi) + \Omega'_u{}^*(\phi; \psi) = 0$$

for any choice of the arguments  $\phi$  and  $\psi$ . This is the *closure* condition for the 2-form  $\Omega$ .

**Step 8:** *the potential  $\theta$ .*

Since we are working on a manifold with simple topology, by the Poincaré lemma we can affirm the existence of a 1-form  $\theta$  such that  $\omega = d\theta$ . In our particular context the 1-form  $\theta$  may be represented as a vector-valued differential operator

$$\theta = \theta(u, u_x, \dots)$$

and its exactness, in operator formalism, may be explicitly written as

$$\Omega = \theta'_u{}^* - \theta'_u \quad (29)$$

where  $\theta'_u$  is, as usual, the Fréchet derivative of the 1-form  $\theta$ . Like in the finite-dimensional case, the operator  $\theta$  can be reconstructed from  $\Omega$  by a quadrature. The formula

$$\theta = - \int_0^1 \Omega_{\lambda u}(\lambda u) d\lambda \quad (30)$$

means that we must apply the matrix differential operator  $\Omega$ , evaluated at the point  $\lambda u$  on the vector  $\lambda u$  itself. Then we must integrate, terms by terms, the resulting vector-valued differential operator  $\Omega_{\lambda u}(\lambda u)$ , depending on  $\lambda$ , on the interval  $[0, 1]$ . Applications of this formula will be given in the next section.

We have finally achieved our goal. By inserting the representation (29) of the 2-form  $\Omega$  in the representation formula (28) of the cocycle  $Q$  we obtain

$$\begin{aligned} Q &= L_Z(P) + P \cdot (\theta'_u{}^* - \theta'_u) \cdot P \\ &= L_Z(P) + L_{P\theta}(P) \\ &= L_{Z+P\theta}(P) \end{aligned}$$

showing that the cocycle  $Q$  is a coboundary. Furthermore we obtain the explicit formula

$$X = Z + P\theta \tag{31}$$

for the potential  $X$  of  $Q$ , as in the finite-dimensional case. The proposition stated at the beginning of this section is thus completely proved.

To prepare the discussion on Dubrovin's conjecture, to be performed in the final section, it remains to understand what relation connects the class of homogeneous bivectors considered by Dubrovin (and described in § 1) to the class of bivectors considered in this section.

**Lemma 7** *The class of Dubrovin's cocycles is strictly contained in the present class of cocycles.*

*Proof.* The point is to show that the homogeneity assumption (together with the cocycle condition) entails the involutivity condition (18) used to define our class of cocycles. To prove this result we exploit the well-known property that, for every cocycle, the bracket  $\{C^a, C^b\}_Q$  is still a Casimir function of  $P$ . This means that

$$Pd(\{C^a, C^b\}_Q) = 0$$

that is

$$\frac{d}{dx} \frac{\delta E^{ab}}{\delta u^l} = 0.$$

Let us write

$$\frac{\delta E^{ab}}{\delta u^l} = A_l^{ab}. \tag{32}$$

By the above condition the functions  $A_l^{ab}$  are constant, and therefore

$$E^{ab} = A_l^{ab} u^l + \frac{d}{dx} K^{ab}.$$

Accordingly

$$\{C^a, C^b\}_Q = \int_0^1 A_l^{ab} u^l dx.$$

The homogeneity condition of Dubrovin entails  $A_l^{ab} = 0$ , since  $E^{ab}$  should have at least degree one. So the combined action of the cocycle and of the homogeneity condition entails the involutivity (18), as required.  $\square$

We finally notice that the vector field  $Z$  and the 1-form  $\theta$  associated with the homogeneous cocycle  $Q$  are themselves homogeneous operators, due to Eq. (23) and Eq. (30). Thus we can end our discussion by stating the following proposition



**Proposition 2** *All cocycles in Dubrovin's class are coboundaries, and their potentials are homogeneous operators.*

In our opinion, this is the deep reason for the validity of Dubrovin's conjecture.

## 4 Three examples

As first example we consider again the homogeneous third-order scalar differential operators (5). We have already shown that they are coboundaries by guessing the form of the vector field  $X$ , inside the class of homogeneous vector fields. Presently we want to rediscover systematically this vector field, by using the previous algorithm. We recall the main steps of this approach.

1. The starting point is the representation formula  $A^t \cdot P + P \cdot A + P \cdot \Gamma \cdot P$ . It allows to identify the matrix  $A$  and the 2-form  $\Gamma$ .
2. The next step is to exploit the pieces of information encoded into the matrix  $A$ . Its columns are exact 1-forms, and their potentials are the Hamiltonians with Lagrangian densities  $h^a$ . They allow to define the transversal vector field  $Z = -h^a Z_a$ .
3. The Fréchet derivative  $Z'$  of this vector field is the last object to be analyzed. Through the second representation formula  $Z' = -A^t + P \cdot R$ , it allows to identify the operator  $R$ , which defines the deformation  $\Omega = \Gamma + R - R^*$  of  $\Gamma$  we are interested in.
4. At this point we compute the potential  $\theta$  of the 2-form  $\Omega$ . The vector field  $X$  is given by the formula  $X = -H^a Z_a + P\theta$ .

For the example at hands, the first representation formula reads

$$\begin{aligned} P_2 = & [2\alpha(u)u_{xx} + \alpha'(u)u_x^2] \frac{d}{dx} + \frac{d}{dx} \cdot [2\alpha(u)u_{xx} + \alpha'(u)u_x^2] \\ & + \frac{d}{dx} \cdot [\beta(u) \frac{d}{dx} + \frac{d}{dx} \cdot \beta(u)] \cdot \frac{d}{dx} \end{aligned}$$

where  $\alpha(u)$  and  $\beta(u)$  are related to the previous coefficients  $E(u)$  and  $G(u)$  according to

$$\begin{aligned} \alpha(u) &= \frac{1}{4}(G(u) - \frac{1}{2}E'(u)) \\ \beta(u) &= \frac{1}{2}E(u). \end{aligned}$$

Since  $P = \frac{d}{dx}$ , the “matrix”  $A$  is simply the scalar function

$$A(u; u_x, u_{xx}) = 2\alpha(u)u_{xx} + \alpha'(u)u_x^2.$$

We recognize in this expression the Euler operator associate with the Lagrangian density

$$h(u; u_x) = -\alpha(u)u_x^2.$$

Consequently the vector field  $Z$  is given by

$$Z(u; u_x) = \alpha(u)u_x^2 = \frac{1}{4}(G(u) - \frac{1}{2}E'(u))u_x^2.$$

Its Fréchet derivatives is

$$Z'_u = -2\alpha(u)u_{xx} - \alpha'(u)u_x^2 + 2\frac{d}{dx}[\alpha(u)u_x]$$

and therefore

$$Z'_u + A^t = 2\frac{d}{dx}[\alpha(u)u_x].$$

In this way we obtain the operator  $R = 2\alpha(u)u_x$ . Since  $R^* = R$ , the 2-form  $\Omega$  is simply given by

$$\Omega = \beta(u)\frac{d}{dx} + \frac{d}{dx} \cdot \beta(u).$$

This form is exact and its potential is given by

$$\theta(u; u_x) = -\beta(u)u_x = -\frac{1}{2}E(u)u_x.$$

It can be computed either by direct inspection or by using the equation

$$\begin{aligned} \theta &= -\int_0^1 \Omega_{\lambda u}(\lambda u) d\lambda \\ &= -\int_0^1 [\beta(\lambda u)\frac{d}{dx}(\lambda u) + \frac{d}{dx}(\beta(\lambda u))] d\lambda. \end{aligned}$$

Finally, the vector field  $X$  is given by

$$X = Z + P\theta = -\frac{1}{2}E(u)u_{xx} + (\frac{1}{4}G(u) - \frac{5}{8}E'(u))u_x^2.$$

It coincides with the vector field already obtained at the beginning of § 3.

As a second example let us consider the non-homogeneous third-order scalar differential operator

$$Q = \frac{d^3}{dx^3} + 2u\frac{d}{dx} + u_x. \quad (33)$$

It is the well-known second Hamiltonian operator of the KdV hierarchy. By writing  $Q$  in the form

$$Q = u \cdot \frac{d}{dx} + \frac{d}{dx} \cdot u + \frac{d}{dx} \left( \frac{d}{dx} \right) \cdot \frac{d}{dx}$$

we identify

$$A(u) = u$$

$$\Gamma = \frac{d}{dx}.$$

Once again, we recognize in  $A$  the Euler operator associate with the lagrangian density

$$h(u) = \frac{1}{2}u^2.$$

Consequently

$$Z = -\frac{1}{2}u^2.$$

Its Fréchet derivative verifies the equation

$$Z' + A^t = -u + u = 0.$$

Hence  $R = 0$ , and  $\Omega = \frac{d}{dx}$ . The potential  $\theta$  of this 2-form is

$$\theta = -\frac{1}{2}u_x.$$

The potential  $X$  associated with the operator  $Q$  is then

$$X = -\frac{1}{2}u^2 - \frac{1}{2}u_{xx}. \quad (34)$$

The final example concerns a non-homogeneous matrix-valued bivector  $Q$ . We consider the pair of Poisson bivectors

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} \quad (35)$$

and

$$Q = \begin{pmatrix} -\frac{d^3}{dx^3} + 2u\frac{d}{dx} + u_x & 3v\frac{d}{dx} + 2v_x \\ 3v\frac{d}{dx} + v_x & \frac{d^5}{dx^5} - \frac{10}{3}u\frac{d^3}{dx^3} - 5u_x\frac{d^2}{dx^2} + (\frac{16}{3}u^2 - 3u_{xx})\frac{d}{dx} + (\frac{16}{3}uu_x - \frac{2}{3}u_{xxx}) \end{pmatrix} \quad (36)$$

associated with the Boussineq hierarchy. It is well-known that  $P$  is a coboundary of  $Q$ . Indeed  $P$  is the Lie derivative of  $Q$  along the vector field

$$X_{-1} : \begin{cases} \dot{u} = 0 \\ \dot{v} = \frac{1}{2} \end{cases}$$

Our aim is presently to show that  $Q$  is a coboundary of  $P$ , and we want to compute explicitly its potential  $X_1$ :

$$Q = L_{X_1}(P).$$

The boring part is the identification of the matrix  $A$  and of the operator  $\Gamma$ . We start from the representation formula

$$\begin{aligned} Q = & E + S \cdot \frac{d}{dx} + \frac{d}{dx} \cdot S + \\ & \frac{d}{dx} \cdot [\Lambda_0 + (\Lambda_1 \cdot \frac{d}{dx} + \frac{d}{dx} \cdot \Lambda_1) + (\Lambda_2 \cdot \frac{d^2}{dx^2} + \frac{d^2}{dx^2} \cdot \Lambda_2) \\ & + (\Lambda_3 \cdot \frac{d^3}{dx^3} + \frac{d^3}{dx^3} \cdot \Lambda_3)] \cdot \frac{d}{dx} \end{aligned} \quad (37)$$

valid for any fifth-order bivector. By comparison with Eq. (36), we obtain

$$\begin{aligned} E = \begin{pmatrix} 0 & \frac{1}{2}v_x \\ -\frac{1}{2}v_x & 0 \end{pmatrix} \quad S = \begin{pmatrix} u & \frac{3}{2}v \\ \frac{3}{2}v & \frac{8}{3}u^2 - \frac{2}{3}u_{xx} \end{pmatrix} \quad \Lambda_0 = 0 \\ \Lambda_1 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{5}{3}u \end{pmatrix} \quad \Lambda_3 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \Lambda_2 = 0 \end{aligned}$$

We notice that  $E$  is a total derivative with respect to  $x$ , guaranteeing that  $\{C^a, C^b\}_Q = 0$ . Since

$$E = \frac{d}{dx} \cdot \begin{pmatrix} 0 & \frac{1}{2}v \\ -\frac{1}{2}v & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2}v \\ \frac{1}{2}v & 0 \end{pmatrix} \cdot \frac{d}{dx} \quad (38)$$

we obtain the first representation formula of the operator  $Q$

$$\begin{aligned} Q &= \frac{d}{dx} \cdot \begin{pmatrix} u & 2v \\ v & \frac{8}{3}u^2 - \frac{2}{3}u_{xx} \end{pmatrix} + \begin{pmatrix} u & v \\ 2v & \frac{8}{3}u^2 - \frac{2}{3}u_{xx} \end{pmatrix} \cdot \frac{d}{dx} \\ &+ \frac{d}{dx} \cdot \begin{pmatrix} -\frac{d}{dx} & 0 \\ 0 & \frac{d^3}{dx^3} - \frac{10}{3}u\frac{d}{dx} - \frac{5}{3}u_x \end{pmatrix} \cdot \frac{d}{dx}. \end{aligned}$$

Replacing the operator  $\frac{d}{dx}$  by the bivector  $P$ , we obtain the second representation formula

$$\begin{aligned} Q &= P \cdot \begin{pmatrix} v & \frac{8}{3}u^2 - \frac{2}{3}u_{xx} \\ u & 2v \end{pmatrix} + \begin{pmatrix} \frac{8}{3}u^2 - \frac{2}{3}u_{xx} & u \\ 2v & \end{pmatrix} \cdot P \\ &+ P \cdot \begin{pmatrix} \frac{d^3}{dx^3} - \frac{10}{3}u\frac{d}{dx} - \frac{5}{3}u_x & 0 \\ 0 & -\frac{d}{dx} \end{pmatrix} \cdot P. \end{aligned}$$

It follows that

$$A = \begin{pmatrix} v & \frac{8}{3}u^2 - \frac{2}{3}u_{xx} \\ u & 2v \end{pmatrix}$$

and

$$\Gamma = \begin{pmatrix} \frac{d^3}{dx^3} - \frac{10}{3}u\frac{d}{dx} - \frac{5}{3}u_x & 0 \\ 0 & -\frac{d}{dx} \end{pmatrix}.$$

At this point we just repeat the usual scheme. We notice that the entries of the columns of  $A$  are the Euler operators associated with the Lagrangian densities

$$\begin{aligned} h^1(u, v) &= uv \\ h^2(u, v; u_x, v_x) &= v^2 + \frac{8}{9}u^3 + \frac{1}{3}u_x^2 \end{aligned}$$

respectively. Then we use the transversal symmetries

$$\begin{aligned} z_1 : & \begin{cases} \dot{u} = 1 \\ \dot{v} = 0 \end{cases} \\ z_2 : & \begin{cases} \dot{u} = 0 \\ \dot{v} = 1 \end{cases} \end{aligned}$$

to build the vector field

$$Z : \begin{cases} \dot{u} = -uv \\ \dot{v} = -v^2 - \frac{8}{9}u^3 - \frac{1}{3}u_x^2. \end{cases}$$

Its Fréchet derivative verifies the equation

$$\begin{aligned} Z' + A^t &= \begin{pmatrix} -v & -u \\ -\frac{8}{3}u^2 - \frac{2}{3}u_x\frac{d}{dx} & -2v \end{pmatrix} + \begin{pmatrix} \frac{8}{3}u^2 - \frac{2}{3}u_{xx} & u \\ 2v & \end{pmatrix} \\ &= \frac{d}{dx} \cdot \begin{pmatrix} 0 & 0 \\ -\frac{2}{3}u_x & 0 \end{pmatrix}. \end{aligned}$$

So the operator  $R$  is given by

$$R = \begin{pmatrix} -\frac{2}{3}u_x & 0 \\ 0 & 0 \end{pmatrix}.$$

Since  $R = R^*$  we finally get  $\Omega = \Gamma$ . Its potential  $\theta$  is:

$$\theta = \begin{pmatrix} \frac{5}{3}uu_x - \frac{1}{2}u_{xxx} \\ \frac{1}{2}v_x \end{pmatrix}.$$

Consequently

$$X_1 : \begin{cases} \dot{u} &= -uv + \frac{1}{2}v_{xx} \\ \dot{v} &= -v^2 - \frac{8}{9}u^3 + \frac{4}{3}u_x^2 + \frac{5}{3}uu_{xx} - \frac{1}{2}u_{xxxx}. \end{cases}$$

Let us finally consider the third vector field

$$2X_0 = [X_{-1}, X_1]$$

It is a conformal symmetry of both Poisson bivectors  $P$  and  $Q$ . Indeed

$$\begin{aligned} L_{X_0}(P) &= \frac{1}{2}P \\ L_{X_0}(Q) &= \frac{1}{2}Q. \end{aligned}$$

Furthermore, the vector fields  $(X_{-1}, X_0, X_1)$  satisfy the commutation relations

$$[X_{-1}, X_0] = X_1 \quad [X_1, X_0] = -X_{-1}.$$

Therefore by the present algorithm we have constructed the  $\mathfrak{sl}(2)$ -subalgebra of the  $\mathcal{W}$ -algebra associated with the Boussineq hierarchy. This remark suggests that the method used in this paper are potentially very useful in analyzing and classifying Poisson pencils on bihamiltonian manifolds.

## 5 Proof of Dubrovin's conjecture

The key idea for proving the conjecture is to reduce the Jacobi identity  $[P_\epsilon, P_\epsilon] = 0$  to a sequence of cohomological equations. This is possible on a manifold of hydrodynamic type due to the results of § 3. The outcome is a peculiar representation of the coefficients of the deformation  $P_\epsilon$  in terms of vector fields.

**Proposition 3** *A sequence of homogeneous vector fields  $X_k$  may be associated with every homogeneous deformations  $P_\epsilon$ , in such a way that the coefficients  $P_k$  of the Taylor expansion of  $P_\epsilon$  are written as iterated derivatives of the given bivector  $P_0$ . To this end consider the Lie derivatives associated with the vector fields, and construct with them the operator*

$$T_k = \sum_{j_1+2j_2+\dots+kj_k=k} \frac{L_{X_k}^{j_k}}{j_k!} \dots \frac{L_{X_1}^{j_1}}{j_1!}$$

*to be referred to as the Schur polynomial of order  $k$  associated with the given sequence of vector fields. Then*

$$P_k = T_k(P_0). \quad (39)$$

*Proof.* Let us first check the formula for  $k = 1$ . We know that the first coefficient  $P_1$  is an homogeneous bivector verifying the cocycle condition  $[P_1, P_0] = 0$ . Hence, by the final proposition of § 3 there exist an homogeneous vector field  $X_1$ , such that  $P_1 = L_{X_1}(P_0)$ . This proves the first case of identity (39).

To prove by induction the remaining cases, we use the identity

$$T_k([P, P]) = \sum_{\substack{j, l=0 \\ j+l=k}}^k [T_j(P), T_l(P)]. \quad (40)$$

It follows from the transformation law

$$\psi_{\epsilon*}([P, P]) = [\psi_{\epsilon*}(P), \psi_{\epsilon*}(P)]$$

with respect to the special one parameter family of local diffeomorphisms

$$\phi_{\epsilon}^{(k)} : M \rightarrow M$$

constructed as follows. First we compose the flows  $(\phi_{t_1}, \dots, \phi_{t_k})$  associated with the vector fields  $(X_1, \dots, X_k)$  so to obtain the multiparameter family of local diffeomorphisms

$$\phi_{t_1, \dots, t_k}^{(k)} = \phi_{t_k} \circ \dots \circ \phi_{t_1}. \quad (41)$$

Then we reduce this family by setting

$$t_j = \epsilon^j. \quad (42)$$

By expanding Eq. (41) in powers of  $\epsilon$ , and by equating the coefficients of  $\epsilon^k$  we obtain exactly Eq. (40).

Assume presently that the representation (39) is true for the first  $n$  coefficients  $(P_1, \dots, P_n)$ . To prove that it is also true for  $P_{n+1}$  we consider Eq. (40) for  $k = n + 1$ . We notice that this equation holds for any choice of the vector fields  $(X_1, \dots, X_{n+1})$ . In particular it holds also for  $X_{n+1} = 0$ . Let us denote by

$$\hat{T}_{n+1} = T_{n+1}|_{X_{n+1}=0}$$

the restriction of the operator  $T_{n+1}$  to the first  $n$  vector fields of the sequence. Then we can write

$$\hat{T}_{n+1}([P_0, P_0]) = \sum_{\substack{j, l=0 \\ j+l=n+1}}^{n+1} [\hat{T}_j(P_0), \hat{T}_l(P_0)]. \quad (43)$$

By assumption  $[P_0, P_0] = 0$ , and

$$P_l = T_l(P_0) = \hat{T}_l(P_0) \quad \forall l = 1, \dots, n.$$

Therefore Eq. (43) becomes:

$$2[P_0, \hat{T}_{n+1}(P_0)] + \sum_{\substack{j, l=1 \\ j+l=n+1}}^n [P_j, P_l] = 0.$$

Let us compare this equation with

$$2[P_0, P_{n+1}] + \sum_{\substack{j,l=1 \\ j+l=n+1}}^n [P_j, P_l] = 0$$

expressing the Jacobi identity  $[P_\epsilon, P_\epsilon] = 0$  at the order  $n+1$  in  $\epsilon$ . It takes the form of a cocycle condition:

$$[P_0, P_{n+1} - \hat{T}_{n+1}(P_0)] = 0.$$

Therefore there exist a vector field  $X_{n+1}$  such that

$$P_{n+1} = L_{X_{n+1}}(P_0) + \hat{T}_{n+1}(P_0) = T_{n+1}(P_0).$$

By induction this proves the representation formula (39) for any  $k$ .  $\square$

To end the proof of Dubrovin's conjecture it is sufficient now to notice that the infinite sequence of identities

$$P_k = T_k(P_0) \tag{44}$$

means that

$$P_\epsilon = \phi_{\epsilon*}(P_0) \tag{45}$$

for the limit  $\phi_\epsilon$  of the sequence of local diffeomorphism  $\phi_\epsilon^{(k)}$  for  $k \rightarrow \infty$ . Indeed, according to the theory of "Lie transform", Eq. (44) are nothing else than the Taylor expansion of Eq. (45) in powers of  $\epsilon$ . We have then obtained a constructive proof of Dubrovin's conjecture. The relation

$$\phi_\epsilon^{(k)} = \phi_{\epsilon^k}^{[X_k]} \circ \dots \circ \phi_\epsilon^{[X_1]} \tag{46}$$

gives the approximation, at order  $k$  of the trivializing map  $\phi_\epsilon : M \rightarrow M$  we were looking for.

## Acknowledgments

We sincerely thank B. Dubrovin for introducing us to the problem of deformation of Poisson manifolds of hydrodynamic type. We also thank G. Falqui for many useful discussion. We finally thank the Istituto Nazionale di Alta Matematica of Rome, who supported a meeting on the geometry of Frobenius manifolds, giving us the occasion to meet all together and discuss the problem.

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